Synchronization on Erdös-Rényi networks

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In this Brief Report, by analyzing the spectral properties of the Laplacian matrix of Erdös-Rényi networks, we obtained the critical coupling strength of the complete synchronization analytically. In particular, for any size of the networks, when the average degree is greater than a threshold and the coupling strength is large enough, the networks can synchronize. Here, the threshold is determined by the value of the maximal Lyapunov exponent of each dynamical unit.

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Recently, it is clear that complex networks give a good description of diverse systems such as the Internet, neural networks, metabolic systems, and power grid [1–3]. Many of the studies focused on the topological properties such as degree distribution, clustering structure, and geographical constraints [4–6]; meanwhile, various dynamics on complex networks are also investigated [7–9]. Underlying those researches is the implication that the topological properties of a network must have some influence on the dynamics taking place on it [10].

As one of the simplest collective dynamics, synchronization of coupled nonlinear oscillators is studied on various networks through numerical experiments as well as analytical methods. Early studies of synchronization are restricted to networks of globally coupled units and simple networks with some symmetry [11–13], but the structure of the real-world network is generally more complex. Recently, Pecora and Carrol obtained the exact results of synchronization for networks with a general structure [14]. In their paper, they promoted a linear stability condition of synchronization on arbitrary coupling networks and the critical coupling strength can be obtained by finding the solution of the master stability equations. The solutions are not explicitly related to the fundamental parameters of the networks such as the network size, the average degree. And the size of networks are generally enormous, to solve the equations usually will cost much time and computer resource. Recently many works have been done to find the shortcuts from the network parameters to the critical coupling strength [15–18] for different network models. Then, it will be helpful to understand how the topological properties of the network affect the synchronization, if one can obtain a direct solution.

In this paper, we study the synchronization on the Erdös-Rényi (ER) network. By analyzing the spectral properties of the Laplacian matrix of the networks, we give an approximate expression of critical coupling strength related only with the network size N and the average degree $\langle k\rangle$. Numerical results are also obtained, and well consistent with our analytical formula.

Consider an ER network of N identical, linearly and diffusively coupled nodes, each pair of nodes being connected with probability p, and the average degree is N_p. Each node is an m-dimensional discrete dynamical system

$$x_{n+1}^i = f(x_n^i) + \frac{\epsilon}{k_i}\sum_{j=1}^{N} \Gamma_{i,j} f(x_n^j), \tag{1}$$

where k_i is the degree of node i, x_n^i is the state of the node i at time n, ϵ is the coupling strength, and Γ is the Laplacian matrix

$$\Gamma_{i,j} = \begin{cases} -k_i, & i=1, \\ 1, & \text{if node } i \text{ connects node } j, \\ 0, & \text{otherwise}. \end{cases} \tag{2}$$

This model can be extended straightforwardly to time-continuous case.

We then rewrite Eq. (1) in the following form

$$x_{n+1}^i = f(x_n^i) + \epsilon\sum_{j=1}^{N} \Lambda_{i,j} f(x_n^j), \tag{3}$$

where $\Lambda_{i,j}=\Gamma_{i,j}/k_i$. The linear stability of synchronization of this system is determined by the eigenvalues of the matrix Λ [14]. The eigenvalues of Λ are real and nonpositive [19], we write them as $-\lambda_k$, and then the eigenvalue equation is

$$\Lambda u_i + \lambda_i u_i = 0. \tag{4}$$

We order the eigenvalues as $0=\lambda_1<\lambda_2\leqslant\cdots\leqslant\lambda_{N-1}\leqslant\lambda_N$. The eigenvalue λ_1 is a simple eigenvalue, since we assume that the network is connected [19].

The synchronization state of this system is stable if

$$\frac{1-e^{-\mu}}{\lambda_2} < \epsilon < \frac{1+e^{-\mu}}{\lambda_N}, \tag{5}$$

where μ is the largest Lyapunov exponent of f [19]. When N is large, and the degree of each node could be regarded as the same: pN, then we have

$$\Lambda \simeq A/(Np) - 1, \tag{6}$$

where A is the adjacency matrix of the network.

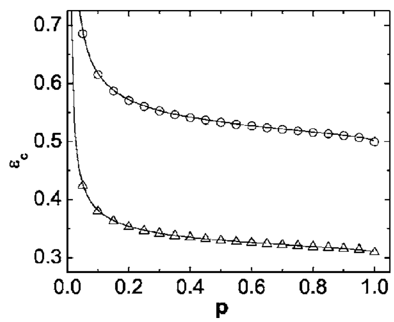

FIG. 1. Critical coupling strength of synchronization on ER networks for different a (a=1.6 for triangles, 2.0 for circles) and different connectivity probability p, as the x axis shows. The curves are the corresponding analytical results. Averaged over 1000 network realizations and initial conditions, N=1000.

We denote the eigenvalues of A as $\bar{\lambda}_N \leq \bar{\lambda}_{N-1} \leq \cdots \bar{\lambda}_2 \leq \bar{\lambda}_1$. From (6), the eigenvalues of Λ are obtained

$$\lambda_i \simeq -\bar{\lambda}_i/(Np) + 1. \tag{7}$$

When $N \to \infty$ the spectral density of the adjacent matrix A (with connecting probability p) converges to a semicircular distribution [20]

$$\rho(\lambda) = \begin{cases} \dfrac{\sqrt{4Np(1-p) - \lambda^2}}{2\pi Np(1-p)}, & \text{if } |\lambda| < 2\sqrt{Np(1-p)}, \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

The largest eigenvalue $\bar{\lambda}_1$ is isolated from the bulk of the spectrum, and it increases with the network size as pN. This means that $\bar{\lambda}_n \simeq -2\sqrt{Np(1-p)}$, $\bar{\lambda}_2 \simeq 2\sqrt{Np(1-p)}$. It is easy to see that

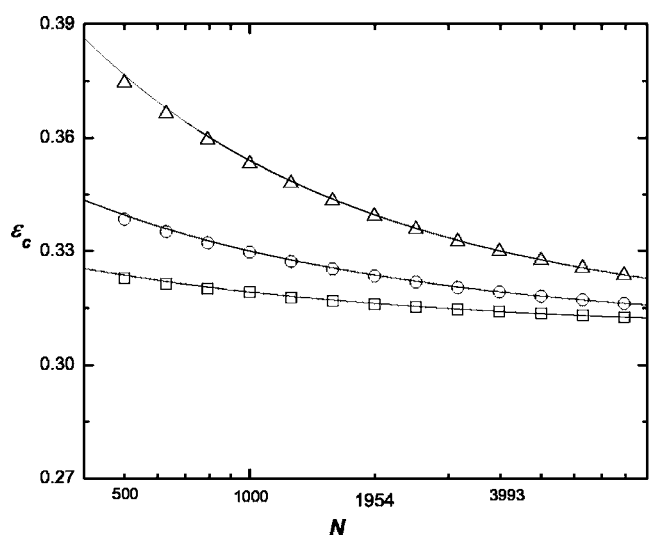

FIG. 2. Critical coupling strength of synchronization for different connectivity probability p (p=0.2 for squares, 0.5 for circles, and 0.8 for triangles). We can see that when $N \to \infty$, networks with different connectivity probability have the same critical coupling value of synchronization, i.e., the value of GCN.

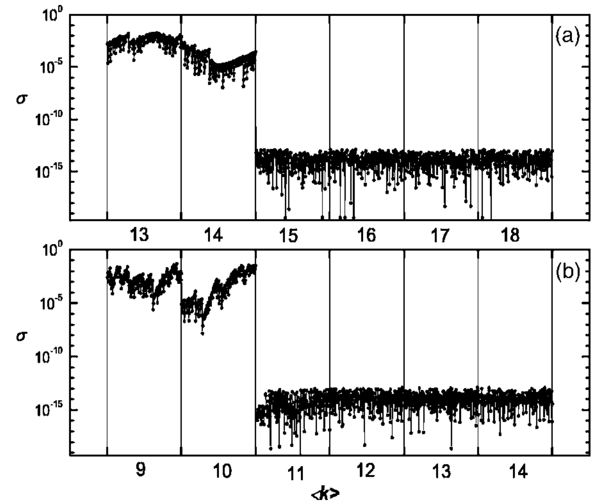

FIG. 3. The fluctuation σ for different $\langle k \rangle$, for N=2000, ϵ=1.0 (a) a=1.9 and (b) a=2.0.

$$\lambda_2 \simeq 1 - 2\sqrt{(1-p)/(Np)},$$

$$\lambda_n \simeq 1 + 2\sqrt{(1-p)/(Np)}. \tag{9}$$

Substituting (9) into (3), the stability condition of synchronization becomes

$$\frac{1 - e^{-\mu}}{1 - 2\sqrt{(1-p)/(Np)}} < \epsilon < \frac{1 + e^{-\mu}}{1 + 2\sqrt{(1-p)/(Np)}}. \tag{10}$$

In order to satisfy this condition, we need

$$\frac{1 + 2\sqrt{(1-p)/(Np)}}{1 - 2\sqrt{(1-p)/(Np)}} < \frac{1 + e^{-\mu}}{1 - e^{-\mu}}. \tag{11}$$

[Since in most relevant cases, the upper limit $(1+e^{-\mu})/(1+2\sqrt{(1-p)/(Np)})$ is greater than 1, so in the following, we just discuss the lower limit $(1-e^{-\mu})/(1-2\sqrt{(1-p)/(Np)})$].

To verify the above analytical formula (10), we also give numerical results. In this paper, we work with the logistic map $f(x)=1-ax^2$ and choose values of a such that the dynamics of individual map is chaotic. Here, the critical coupling strength for different connectivity probability and different a is shown by both numerical simulation and analytical computation in Fig. 1. We can see that our formula gives a precise result.

From our analytical expression of the critical coupling strength, we can strictly deduce some conclusions.

(1) $p \to 1$, Eq. (10) becomes

$$\epsilon > 1 - e^{-\mu}, \tag{12}$$

consistent with the result of globally coupled network (GCN) [21].

(2) Keep p fixed, and let $N \to \infty$, Eq. (10) can also be rewritten as

$$\epsilon > 1 - e^{-\mu},$$

which implies that in this limit, the behavior of this system would be equivalent to that of a GCN.

Under the same limit condition, the mean-field approximation holds, Eq. (1) can be rewritten as

$$x_{n+1}^{i}=f(x_{n}^{i})+\frac{\epsilon}{N}\sum_{j=1}^{N}[f(x_{n}^{j})-f(x_{n}^{i})],$$

which is just the dynamics of GCN. We obtain the same result as deduced by the above analytical method. This result was obtained by simulation in [22]. In Fig. 2, we show that in the limit $N\rightarrow\infty$, networks with different connectivity probability p all approach the critical value of GCN.

(3) Let $k=Np$, $\epsilon=1$, Eq. (10) becomes

$$\frac{1-e^{-\mu}}{1-2\sqrt{1/k-1/N}}<1. \tag{13}$$

When $N\gg k$, we obtain

$$\frac{1-e^{-\mu}}{1-2\sqrt{1/k}}<1, \tag{14}$$

or

$$k>K=4e^{2\mu}, \tag{15}$$

which implies that for ER random networks, one can have chaotic synchronization for arbitrary system size, if the coupling is strong enough, i.e., $\epsilon=1$ and if the average degree is larger than some threshold determined by the value of the maximal Lyapunov exponent of the individual dynamics. In [19], the authors gave an experimental result for this. They said that for the quadratic map [i.e., $f(x)=1-ax^2$], when $a=2.0$, $\epsilon=1$, and average degree is greater than 16, or when $a=1.9$, $\epsilon=1$, and the average degree is greater than 12, the system can synchronize for arbitrary large size N. Here we can analytically obtain above results. When $a=2.0$ the largest Lyapunov exponent is ln 2, according to (15), the critical average degree K is 16, and when $a=1.9$ the largest Lyapunov exponent is 0.55, so the corresponding K is 12, consistent with the experimental result [19]. Here we show the fluctuation σ as a function of $\langle k\rangle$ for $N=2000$, $\epsilon=1.0$, $a=1.9$ [Fig. 3(a)], and $a=2.0$ [Fig. 3(b)]. In each grid corresponding to a particular $\langle k\rangle$ value, we plot 2000 steps of σ after initial transients.

In conclusion, we have derived accurate analytical results for the threshold of synchronization in the ER random network. Our results also show that once the average degree is greater than a particular value, the network of arbitrary size can always be synchronous with strong enough coupling.

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